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Exact (1 + 1)-Dimensional Solutions of Discrete Planar Velocity Boltzmann Models

Henri Cornille[1]

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For discrete velocity Boltzmann models we have found $(1+1)$-dimensional shock waves and periodic solutions that are rational solutions with two exponential variables $\exp(\gamma_i x + \rho_i t)$ (space x, time t). These exact solutions are sums of two rational solutions, each with one exponential variable (similarity solutions). We study the planar velocity models and explicitly write the results for the square 4-velocity and the hexagonal 6-velocity models introduced by Gatignol.

KEY WORDS: Kinetic theory; discrete Boltzmann models.

1. INTRODUCTION

There is interest in discrete Boltzmann models because one hopes to extract useful information in both kinetic theory and fluid mechanics. Recently[1] for the 2-velocity Illner[2] model and the 6-velocity Broadwell[3] model we have obtained exact $(1+1)$-dimensional (space x, time t) solutions. Here we extend these results for a planar hexagonal 6-velocity model.[4] There exist models with velocities on a line, in a plane, or in three-dimensional space.

The simplest models are the 2-velocity ones on a line (Carleman, McKean,[5] Illner, etc.). If we except a completely solvable model,[6] the only exact solutions known[7] were the one-dimensional similarity ones or solutions deduced from ordinary differential equations. These 2-velocity models do not satisfy momentum conservation. Among the other models,[8] the Broadwell model with six velocities in three-dimensional space is the most popular. It has only three or four different densities if the solutions depend upon one spatial coordinate.

[1] Service de Physique Théorique, CEN-Saclay, 91191 Gif-sur Yvette, France.

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In a mathematical analogy with the $(1+1)$-dimensional integrable case, let us call "multisolitons" the rational solutions with exponential variables $u_i = d_i \exp(\gamma_i x + \rho_i t)$. For the discrete models, the "solitons" are the similarity shock-wave solutions and the "bisolitons" either the superposition of two similarity shock waves or periodic solutions.

For the discrete models let us consider for the densities N_i (associated to the discrete velocity $\mathbf{v}_i$) the so-called "planar shock-wave" solutions. These are rational solutions of the type $N_i = n_{0i} + n_i/\Delta$, $\Delta = 1 + u$, $u = d \exp(\gamma x + \rho t)$ $(n_{0i}, n_i, d, \gamma, \rho$ are constants). They have the same analytical structure as the kinks of the integrable systems. They are in fact one-dimensional similarity solutions. Seeking really $(1+1)$-dimensional rational solutions for the discrete models with velocity on a line or in the three-dimensional space, we have found[1] that they are simply a linear superposition of two such similarity shock-wave solutions: $N_i = n_{0i} + \sum n_{ji}/\Delta_j$, $\Delta_j = 1 + u_j$, $u_j = d_j \exp(\gamma_j x + \rho_j t)$, $j = 1, 2$ $(n_{0i}, n_{ji}, d_j, \gamma_j, \rho_j$ are constants). This is an astonishing result, because the discrete models are really nonlinear. However, they are in fact weakly nonlinear and they contain linear differential relations.

Here we consider the third class of discrete models, with the velocities in a plane, and, as in the previous cases, we find that the $(1+1)$-dimensional rational solutions are still the superposition of two similarity shock-wave solutions. Here we construct such solutions and discuss their physical properties. Trying to understand the origin of this particular class of rational solutions in $1+1$ dimensions, we remark that the discrete models (for more than two discrete velocities) always have at least two independent linear differential relations (conservations laws of mass and momentum). In the Appendix we show that this gives stong restrictions for the possible multiexponential rational solutions. The study cannot be complete. However, we have not found other solutions, involving several exponentials, than the linear superposition of similarity solutions.

Here we consider $2r$ $(2r = 4, 6, 8,...)$ velocity models in a plane,[4] introduced by Gatignol, with $\mathbf{v}_i + \mathbf{v}_{i+r} = 0$, $|\mathbf{v}_i| = 1$. In the x, y spatial coordinate plane we choose $\mathbf{v}_0$ $(\mathbf{v}_r)$ to be along the positive (negative) x axis, the angle between $\mathbf{v}_0$ and $\mathbf{v}_i$ being $i\pi/r$, with $c_i = \cos(i\pi/r)$. For each velocity $\mathbf{v}_i$ we associate a density N_i. For the solutions which depend only on x the momentum $\boldsymbol{J} = \sum N_i \mathbf{v}_i$ has one x component and necessarily $N_i = N_{2r-i}$, $i = 1,..., r-1$, leaving only $r+1$ different densities N_i, $i = 0,..., r$, among the $2r$ ones. The equations for these models are

$$L_i N_i = \mathrm{Col}_i, \qquad L_i = \partial_t + c_i \partial_x$$

$$\mu\, \mathrm{Col}_i = -(r-1)\, N_i N_{i+r} + \sum_{m=1}^{r-1} N_{i+m} N_{i+m+r}$$

with lost and gain terms in the collision term Col_i. The mass M and momentum J conservation laws are satisfied:

$$M = N_0 + N_r + 2 \sum_1^{r-1} N_i, \qquad J = N_0 - N_r + 2 \sum_1^{r-1} c_i N_i$$
$$M_t + J_x = 0, \qquad J_t + \left(N_0 + N_r + 2 \sum_1^{r-1} c_i^2 N_i\right)_x = 0 \tag{1.2}$$

For the study of (1.1) it is convenient to distinguish between odd and even r values

$$L_i N_i = L_{r-i} N_{r-i} = \mathrm{Col}_i, \qquad i = 0, \dots, q-1, \qquad \text{for} \quad r = 2q \text{ or } r = 2q-1$$
$$L_0 N_0 + L_q N_q + \sum_1^{q-1} L_i N_i = 0, \qquad r = 2q \tag{1.3}$$
$$0 = L_0 N_0 + \sum_1^{q-1} L_i N_i, \qquad r = 2q-1$$

In the last relation if we replace $L_j N_j$ by Col_i, we obtain a linear relation between the collision terms. The important point is that besides the two linear conservation laws, for $r > 2$ other linear differential relations also exist.

Let us consider a linear superposition of similarity shock waves $N_i = n_{0i} + \sum_j n_{ji}/\Delta_j$, $\Delta_j = 1 + d_j \exp(\gamma_j x + \rho_j t)$, which we substitute into (1.1). On both sides of $L_i N_i = \mathrm{Col}_i$ we find terms proportional to Δ_j^{-1} and Δ_j^{-2}, which give the relations of the jth component similarity shock wave. In addition, in Col_i we have terms $(\Delta_j \Delta_{j'})^{-1}$, $j \neq j'$, which must vanish and represent the compatibility condition between the jth and the j'th components. We have

$$-(r-1)(n_{ji} n_{j'i+r} + n_{j'i} n_{ji+r}) + \sum_1^{r-1} (n_{ji+m} n_{j'i+m+r} + n_{j'i+m} n_{ji+m+r}) = 0 \tag{1.4}$$

These superpositions of similarity shock waves are solutions if we have verified (i) the compatibility of the constraints (1.4) and (ii) the positivity of the solutions. We discuss the simplest 4- and 6-velocity models:

(i) For $r = 2$, the 4-velocity model (1.1)–(1.3) gives $N_3 = N_1$ and

$$N_{0t} + N_{0x} = N_{2t} - N_{2x} = -N_{1t} = N_1^2 - N_0 N_2 \tag{1.5}$$

This model is studied in Section 2. Except for a factor 2 in N_{1t}, the equations are the same as those of the 6-velocity Broadwell model with three different densities.[1]

There exist positive, physically acceptable solutions of the three following types: similarity shock waves, superpositions of two similarity shock waves, and periodic solutions. Perhaps one of the most interesting properties of the discrete models is to provide explicit kinetic solutions of shock waves. We study the infinite-Mach shock solutions provided by the similarity shock-wave solutions. For an infinite-strength shock the upstream temperature is $T=0$. We find explicit solutions such that the total mass $M=N_0+N_2+2N_1$ ratio R across the shock is larger than 1, while the modulus of the sound speed is less than 1. For a superposition of two similarity shock waves we have obtained solutions with an almost infinite strength at fixed t. Then the upstream temperature is of the order of ε, with ε arbitrarily small but finite. The total mass ratio R across the shock is a positive number and as illustration we report a numerical example with $R \simeq 3$. The physical properties of the exact solutions of this model are similar to those[1] of the 6-velocity Broadwell model.

(ii) For $r=3$, the 6-velocity model, which has four different densities, is studied in Sections 3–5. We have $N_5=N_1$ and $N_4=N_2$ and (1.3) becomes for $\mu=2$

$$N_{0t}+N_{0x}=N_{3t}-N_{3x}=-2N_{1t}-N_{1x}=-2N_{2t}+N_{2x}=N_1N_2-N_0N_3 \tag{1.6}$$

with the two conservation laws

$$\begin{gathered} M=N_0+N_3+2(N_1+N_2), \qquad J=N_0-N_3+N_1-N_2 \\ M_t+J_x=0, \qquad J_t+[N_0+N_3+(N_1+N_2)/2]_x=0 \end{gathered} \tag{1.7}$$

We have obtained similarity shock waves, superpositions of two similarity shock waves, and periodic solutions that are positive, physically acceptable solutions.

For both models (1.5) and (1.6) the exact solutions are of the following type:

$$N_i^{(\mathrm{I})}=n_{0i}+n_i/\Delta, \qquad N_i^{(\mathrm{II})}=n_{0i}+n_{ji}/\Delta_j, \qquad N_i^{(\mathrm{III})}=n_{0i}+2\,\mathrm{Re}(n_i/\Delta) \tag{1.8}$$

For $N_i^{(\mathrm{I})}$, the similarity shock waves, $\Delta=1+d\,\exp(\gamma x+\rho t)$ and $n_{0i}, n_i, d>0, \gamma$, and ρ are real; for $N_i^{(\mathrm{II})}$, the superposition of two similarity shock waves, $\Delta_j=1+d_j\,\exp(\gamma_j x+\rho_j t)$ and $n_{0i}, n_{ji}, d_j>0, \gamma_j$, and ρ_j are real. For the periodic solutions $N_i^{(\mathrm{III})}$, we still have $\Delta = 1 + d\exp(\gamma x + \rho t)$, but n_i, $\gamma = i\gamma_{\mathrm{I}}$, $\rho = \rho_{\mathrm{R}} + i\rho_{\mathrm{I}}$, and d are complex, while n_{0i} are real.

The method for the determination of the exact solutions is the same as the previous one.[1] When the N_i are introduced into the discrete models equations, we obtain relations between the parameters. We must have more parameters than independent relations and the difference gives the number of free parameters. We define $y = n_0/n_r$ [$n_r = n_2$ for (1.5) and $n_r = n_3$ for (1.6)] and choose y as the first free parameter, while some n_{0i} are the others. Then we introduce intermediate parameters, which are the ratios of $n_i, i \neq 0, r; \gamma, \rho$ (or $n_{ji}, i \neq 0, r; \gamma_j, \rho_j$) by n_r (or n_{jr}). These intermediate parameters are functions of y alone. The crucial point is the determination of n_r as a function of the free parameters. Once n_r is obtained, we go back to the original parameters, multiplying the intermediate ones by n_r. This is exactly the method followed for the similarity solutions. For the superposition of two similarity shock waves we have in addition two y_j, $j = 1, 2$, corresponding to the two components. The compatibility condition (1.4) gives the relation between y_1 and y_2; we choose $y = y_1$ and this compatibility condition determines y_2. For the periodic solutions the two y_j are complex conjugate, $y_2 = y_1^*$, we still choose $y = y_1$, $|y|$ being the free parameter, while the compatibility condition gives the phase of $y = |y| \exp(iz)$.

In Section 3 we construct the similarity shock-wave solutions of (1.6) and study more particularly the positive solutions corresponding to infinite-strength shocks. It is found that the ratio of the total mass across the shock is larger than 9.

In Section 4 we build up the superposition of two similarity shock waves. We focus our interest on the positive solutions with an almost infinite-strength shock. We find that the total mass ratio across the shock is larger than 17 and as illustration present a numerical example. These solutions have three absolute Maxwellians instead of two for the similarity solutions. The supplementary Maxwellian is the equilibrium state.

In Section 5 we construct the periodic solutions, which, due to $\gamma_{\mathrm{R}} = 0$, have one more relation. The positive physical solutions represent propagating and damped waves. We present two numerical examples: one with many oscillations and a weak damping and the other with a strong damping and few oscillations.

2. EXACT SOLUTIONS FOR THE 4-VELOCITY MODEL

As said in the introduction, except for a factor 2, Eqs. (1.5) are those of the Broadwell model. As a simple pedagogical example, we make explicit here the method leading to the determination of the solutions, while in other sections we shall omit some details. Although positive periodic solutions exist for this model, they are not considered for simplicity; the interested reader should refer to Ref. 1. There exists an invariance property for (1.5): $x \leftrightarrow -x$, $N_0 \leftrightarrow N_2$, $N_1 \leftrightarrow N_1$, which simplifies the study of the different cases.

2.1. Similarity Shock Waves

We have

$$N_i = n_{0i} + n_i/\Delta, \qquad \Delta = 1 + d\exp(\gamma x + \rho t), \qquad d > 0$$

We substitute the ansatz N_i into (1.5), and write that the coefficients of Δ^{-1} and Δ^{-2} are opposite, while the constant is zero. We find five relations among the eight parameters n_{0i}, n_i, γ, ρ:

$$n_0(\rho + \gamma) = n_2(\rho - \gamma) = -n_1\rho = n_1^2 - n_0 n_2 = n_{00}n_2 + n_{02}n_0 - 2n_{01}n_1$$
$$n_{01}^2 = n_{00}n_{02} \qquad (2.1)$$

We choose the ratio $n_0/n_2 = y$ and n_{00} and n_{02} as the three free parameters and want to express the other parameters in terms of the free ones. The last relation (1.1) gives n_{01}. For the others we introduce intermediate parameters $\bar{n}_1$, $\bar{\rho}$, and $\bar{\gamma}$, which are the ratios of n_1, ρ, and γ by n_2 and which depend on y alone:

$$\bar{n}_1 = -2y/(1+y), \qquad -\bar{\rho} = \bar{n}_1 + (1+y)/2, \qquad \bar{\gamma} = \bar{\rho}(1-y)/(1+y) \qquad (2.2)$$

We can express n_2 as a function of the free parameters:

$$-n_2 = (1+y)[4n_{01}y + (1+y)\,n_{00} + yn_{02}]/y(1-y)^2 \qquad (2.3)$$

Finally, we come back to the original parameters: $n_0 = yn_2$, $n_1 = \bar{n}_1 n_2$, $\rho = \bar{\rho} n_2$, $\gamma = \bar{\gamma} n_2$. As in Ref. 1, we can find the intervals of the free parameters leading to positive solutions.

We restrict our study to the infinite-Mach shock, for which the upstream temperature is zero or $n_{02} = 1$, $n_i = 0$ for $i \neq 2$ (the other possibility, $n_{0i} = \delta_{i0}$, can be deduced using the above mentioned invariance property). The expression for n_2 in (2.3) is simplified and, assuming $y < -1$, we find that the densities N_i and the total mass M are positive,

$$N_0 = -y[(1+y)/(1-y)]^2/\Delta, \qquad N_2 = 2y(1+y)/\Delta(1+y)^2 \qquad \gamma = (1-y)/2$$
$$N_2 = 1 - [(1+y)/(1-y)]^2/\Delta, \qquad M = 1 - (1+y)/\Delta \qquad y < -1$$
$$(2.4)$$

Further, the modulus of the sound speed $\rho/\gamma = (1+y)/(1-y)$ is less than 1. Finally, the ratio R of the total mass across the shock is $R = M(\xi = \gamma x + \rho t \to -\infty)/M(\xi \to \infty) = -y$ larger than 1.

2.2. Sum of Two Similarity Shock Waves or (1+1)-Dimensional Shock Waves

Here $N_i = n_{0i} + \sum n_{ji}/\Delta_j$, $\Delta_j = 1 + d_j \exp(\gamma_j x + \rho_j t)$, $d_j > 0$.

We substitute the ansatz into (1.5) and from the coefficients of Δ_j^{-1}, Δ_j^{-2}, and constant we find the jth component similarity relation (2.1), which we rewrite in another way:

$$n_{01}^2 = n_{00} n_{02}, \qquad n_{j1}(n_{j0} + n_{j2}) + 2n_{j0} n_{j2} = 0, \qquad \rho_j + n_{j1} + (n_{j0} + n_{j2})/2 = 0$$
$$\gamma_j(n_{j2} + n_{j0}) = \rho_j(n_{j0} - n_{j2}), \qquad n_{j1}^2 - n_{j0} n_{j2} = 2n_{01} n_{j1} - n_{00} n_{j2} - n_{j0} n_{02} \tag{2.5}$$

In addition, the coefficient of $(\Delta_j \Delta_{j'})^{-1}$, $j \neq j'$, which must be zero, gives the compatibility condition between the jth and the j'th components,

$$2n_{j1} n_{j'1} = n_{j0} n_{j'2} + n_{j2} n_{j'0} \tag{2.6}$$

For the determination we still define $y_j = n_{j0}/n_{j2}$ and the intermediate parameter $\bar{n}_{j1} = n_{j1}/n_{j2} = -2y_j/(1-y_j)$, which we substitute into (2.6). We find a polynomial of the second order:

$$y_j^2 + y_j(y_{j'}^2 - 6y_{j'} + 1)/(y_{j'} + 1) + y_{j'} = 0 \tag{2.6'}$$

If we only have two components $j = 1, 2$, then (2.5)–(2.6′) give 10 relations among 13 parameters and we still have three free parameters. We choose n_{00}, n_{02}, and, for instance, $y_1 = n_{10}/n_{12}$ as the free parameters. First from (2.6′) we deduce y_2 from y_1

$$2y_2(1+y_1) = -B \pm [B^2 - 4y_1(1+y_1)^2]^{1/2}, \qquad B = y_1^2 - 6y_1 + 1 \tag{2.7}$$

Second, we follow the method of Section 2.1: n_{j2} is a function of the free parameters

$$-n_{j2} = (1+y_j)(4n_{01} y_j + (1+y_j) n_{00} + y_j n_{02})/y_j(1-y_j)^2 \tag{2.8}$$

Third, we determine $n_{j0} = y_j n_{j2}$ and $n_{j1} = \bar{n}_{j1} n_{j2}$. Finally, we notice from (2.5) that ρ_j and γ_j can be deduced once the n_{ji}, $i = 0, 1, 2$, $j = 1, 2$, are known.

As for the Broadwell modell model, positive solutions exist; however,

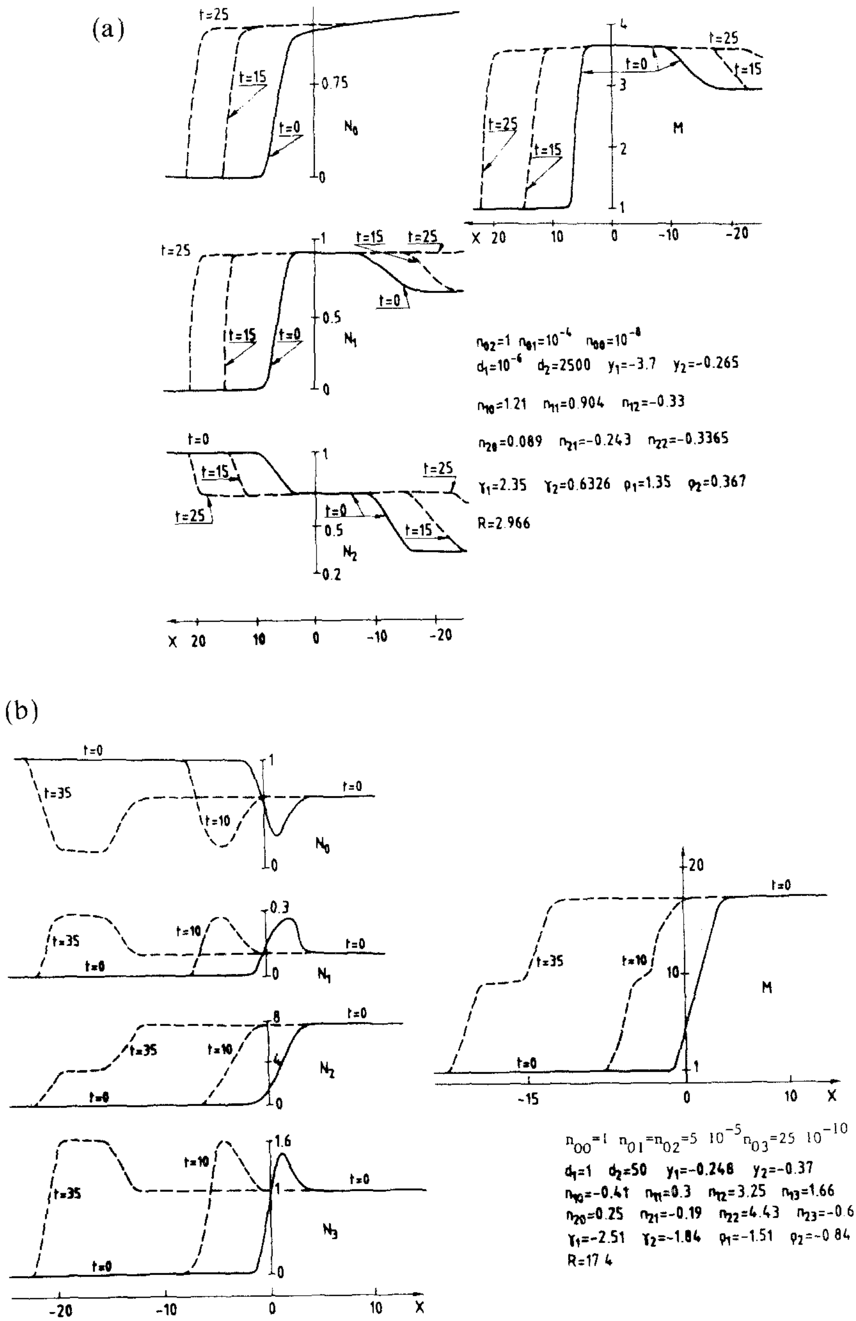

Fig. 1. Sum of two similarity shock waves: (a) 4-velocity model; (b) 6-velocity model.

a complete analytical study is not possible and we must use a computer. First we choose the free parameters such that the asymptotic limits $|x| \to \infty$ of the N_i are positive. Then we use the parameters d_j, $j = 1, 2$, in such a way that the positivity be satisfied at $t = 0$.

Can we have three similarity components? In that case the three parameters y_j, $j = 1, 2, 3$, must satisfy the three relations (2.6)–(2.6′). It turns out that necessarily one of the y_j must be equal to -1 and then the relations (2.5) or (2.1) for that component lead to an impossibility.

At $t = 0$, these $(1+1)$-dimensional solutions look like the similarity ones with two upstream and downstream limits. However, when t increases, we observe deformations of the profils and a third limit occurs, corresponding to the Maxwellian relaxation state.

We can define "almost infinite-strength shock" (see Section 4), for which T is of the order of ε and ε is arbitrarily small but fixed. We can always manage the d_j constants in the denominators Δ_j such that positive densities N_i exist. The ratio of the total mass across the shock can have any finite constant value and can even be very small. For such a shock we have at the upstream $n_{02} = 1$, while the n_{00} and n_{01} are of the order of ε fixed. We notice that in (2.7), when y_1 is close to -1, one of the two y_2 is close to zero. Using the relations (2.4) for the two components, we see that in this limiting case, $y_1 \to -1$, the two γ_j are positive and M has the two limits 1 and 0^+ when $|x| \to \infty$.

As an illustration, in Fig. 1a we plot the relaxation curves for a shock with total mass ratio across the shock $R \simeq 3$.

3. SIMILARITY SHOCK WAVES FOR THE 6-VELOCITY MODEL

Here $N_i = n_{0i} + n_i/\Delta$, $\Delta = 1 + d\exp(\gamma x + \rho t)$, $d > 0$.

There exists for the system (1.6) an invariance property that allows a simplification of the study: if $x \leftrightarrow -x$, then $N_0 \leftrightarrow N_3$ and $N_1 \leftrightarrow N_2$. Substituting the ansatz N_i into (1.6), we find, from the coefficients of Δ^{-1}, Δ^{-2}, and constant, six relations among the ten parameters n_i, n_{0i}, γ, ρ:

$$\begin{aligned} n_0(\gamma+\rho) = n_3(\rho-\gamma) &= -n_1(2\rho-\gamma) = n_2(-2\rho+\gamma) \\ &= n_1 n_2 - n_0 n_3 = n_{00} n_3 + n_{03} n_0 - n_{02} n_1 - n_{01} n_2 \qquad (3.1) \\ n_{01} n_{02} = n_{00} n_{03} & \end{aligned}$$

There exist four arbitrary parameters, chosen to be the ratio $y = n_0/n_3$ and three n_{0i}. The last relation (3.1) gives the fourth n_{0i}. It is useful to replace four relations (3.1) by equivalent ones,

$$n_1(n_0+3n_3) = -2n_0n_3, \qquad n_2(3n_0+n_3) = -2n_0n_3$$
$$-4\rho = n_1+n_2+2(n_0+n_3), \qquad \gamma(n_0+n_3) = \rho(n_3-n_0) \tag{3.1'}$$

As in Section 2, we define intermediate parameters $\bar{n}_1$ and $\bar{n}_2$, which are the ratios of n_1 and n_2 by n_3 and are functions of y alone: $\bar{n}_1 = -2y/(y+3), \bar{n}_2 = -2y/(1+3y)$. The same property holds for the ratios ρ/n_3 and γ/n_3. Then we can express n_3 as a function of the four arbitrary parameters:

$$n_3 = -[(n_{00}+yn_{03})(1+3y)(3+y)+2yn_{02}(1+3y) + 2yn_{01}(3+y)]/[3y(1+y)^2] \tag{3.2}$$

It follows that the four n_i are functions of the four arbitrary parameters: $n_0 = n_3 y$, $n_1 = \bar{n}_1 n_3$, and $n_2 = \bar{n}_2 n_3$. Finally, the last two relations (3.1′) give ρ and γ when the n_i are determined.

For the construction of positive solutions, looking at the limits $|x| \to \infty$, we see that both n_{0i} and $n_{0i}+n_i$ must be positive. It follows from $\Delta \geqslant 1$ that if these conditions are satisfied, then $N_i\Delta = n_{0i}\Delta + n_i \geqslant n_{0i}+n_i$ are positive. The above invariance property is found here with the transform $y \leftrightarrow y^{-1}$, $n_{00} \leftrightarrow n_{03}, n_{01} \leftrightarrow n_{02}$, for which we verify that $n_0 \leftrightarrow n_3$, $n_1 \leftrightarrow n_2$, $\rho \leftrightarrow \rho$, and $\gamma \leftrightarrow -\gamma$.

We discuss the possibility of infinite-strength shocks with $T=0$ upstream. From the definition $dMT = \sum N_i(\mathbf{v}_i - \langle \mathbf{v} \rangle)^2 = (M^2 - J^2)\, M$ (d is the dimension and $\langle \mathbf{v} \rangle$ the mean velocity), it follows that $\mathbf{v}_i = \langle \mathbf{v} \rangle$. This happens only if $n_{0i} = \delta_{i3} n_{03}$ (or if $n_{0i} = \delta_{i0} n_{00}$, which can be deduced, using the first case, from the above invariance).

When $n_{0i} = \delta_{i3}$, the expression (3.2) for n_3 is simplified: $n_3 = -(1+3y)(3+y)/3(1+y)^2$ and for $y < -3$, the densities and the total mass are positive. We have

$$\frac{-N_0}{3+10y+3y^2} = \frac{N_1}{2+6y} = \frac{N_2}{6+2y} = \frac{y(1-N_3)}{3+10y+3y^2} = \frac{y}{3\Delta(1+y)^2}$$
$$M = 1-(1-y)^2/\Delta(y+1), \qquad 2\rho = 1+y, \qquad 2\gamma = 1-y \tag{3.3}$$

Always for $y < -3$, the modulus of the sound speed satisfies the inequality $1/2 < |\rho/\gamma| < 1$, while the ratio $R = y(3-y)/(1+y)$ of the total mass across the shock is larger than 9. [For the $n_{0i} = \delta_{i0}$ case, we apply the transform $y \to y^{-1}$ and find the positivity for $-1/3 < y < 0$, sound speed $1/2 < \rho/\gamma < 1$, and ratio across the shock $R = (3y-1)/y(y+1) > 9$.] Trying to understand the origin of the limiting value of 9, we look at the degenerate case $y = -3$. We obtain $N_0 = N_2 = 0$, $N_3 = 1$, and only

$N_1 = 4[1 + \exp(2x - t)]^{-1}$ is nontrivial, leading for M to the limit $1 + 2(4)$. We remark that this limit, 9, can also be obtained by requiring both $M > 0$ (i.e., $y < -1$ or $0 < y < 3$) and that the modulus of the sound speed $|\rho/\gamma| = |(1+y)/(1-y)|$ be less than the velocity of the flow field, which is 1 ($y < -1$). Then, for $y < -1$ we get $R > 9$. The important point is that the limiting value of 9 can be obtained from the macroscopic quantities.

It seems worth comparing with the values given by the continuous Boltzmann equation for a Maxwellian:

$$M(2\pi T)^{-d/2} \exp\left\{-\left[(\mathbf{v}_1 - \langle \mathbf{v}_1 \rangle)^2 + \sum_2^d v_i^2\right]\Big/ 2T\right\}$$

in d dimensions. Applying the Rankine–Hugoniot jump conditions for a steady shock wave in one dimension, we find

$$\begin{gathered} M\langle \mathbf{v}_1 \rangle = \text{const}, \qquad M\langle \mathbf{v}_1 \rangle^2 + MT = \text{const}, \\ M < \mathbf{v}_1 >^3 + (d+2)\, MT \langle \mathbf{v}_1 \rangle = \text{const} \end{gathered} \tag{3.4}$$

These relations are consequence of the conservation laws of mass, momentum, and energy. We can relate the upstream and downstream macroscopic values. Assuming $T = 0$ upstream and calling R the ratio of the masses across the shock, we obtain from (3.4) the relation $R^2 - (d+2)\, R + d + 1 = 0$ or $R = d + 1$ ($R = 3$ if $d = 2$). As a final remark, we notice that the models (1.5) and (1.6) satisfy only mass and momentum conservation laws.

4. (1+1)-DIMENSIONAL SHOCK WAVES FOR THE 6-VELOCITY MODEL

Here $N_i = n_{0i} + \sum n_{ji}/\Delta_j$, $\Delta_j = 1 + d_j \exp(\gamma_j x + \rho_j t)$, $d_j > 0$.

We substitute the ansatz into (1.6), and from the coefficients of constant, Δ_j^{-1}, and Δ_j^{-2} we find the jth component similarity relations (3.1), (3.1′),

$$\begin{gathered} n_{j1}(n_{j0} + 3n_{j3}) = -2n_{j0}n_{j3}, \qquad n_{j2}(3n_{j0} + n_{j3}) = -2n_{j0}n_{j3} \\ -4\rho_j = n_{j1} + n_{j2} + 2(n_{j0} + n_{j3}) \\ \gamma_j(n_{j0} + n_{j3}) = \rho_j(n_{j3} - n_{j0}) \end{gathered} \tag{4.1}$$

$$\begin{gathered} n_{00}n_{j3} + n_{03}n_{j0} - n_{02}n_{j1} - n_{01}n_{j2} = -n_{j1}(2\rho_j + \gamma_j) \\ n_{01}n_{02} = n_{00}n_{03} \end{gathered}$$

In addition, the vanishing of the coefficient of $(\Delta_j \Delta_{j'})^{-1}$, $j \neq j'$, gives the compatibility condition between the jth and the j'th components:

$$n_{j1} n_{j'2} + n_{j2} n_{j'1} = n_{j0} n_{j'3} + n_{j3} n_{j'0} \tag{4.2}$$

We follow the same method as for the 4-velocity model. We define $y_j = n_{j0}/n_{j3}$, $\bar{n}_{ji} = n_{ji} = n_{ji}/n_{j3}$ for $i = 1, 2$, which are functions of the y_j alone. We rewrite the compatibility condition:

$$\bar{n}_{j1} = -2y_j/(y_j + 3), \qquad \bar{n}_{j2} = -2y_j/(3y_j + 1), \qquad \bar{n}_{j1} \bar{n}_{j'2} + \bar{n}_{j2} \bar{n}_{j'1} = y_j + y_{j'} \tag{4.2'}$$

The compatibility condition becomes $F(y_j, y_{j'}) = 0$, a cubic polynomial in y_j with $y_{j'}$ cubic coefficients. For a superposition of two similarity shock waves we have one polynomial, three polynomials for a superposition $j = 1, 2, 3$, and so on. Although for some $y_{j'}$ intervals $F(y_j, y_{j'}) = 0$ can have three roots, it turns out that we cannot have a superposition with more than two similarity solutions. We define $S = y_1 + y_2$ and $P = y_1 y_2$, and the relation (4.2′) becomes

$$\begin{gathered} P^2 A_1 + P A_2 + A_3 = 0 \\ A_1 = 3S - 8, \qquad A_2 = 10S^2 + 14S - 8, \qquad A_3 = 3S^3 + 10S^2 + 3S \end{gathered} \tag{4.2''}$$

For a superposition of two similarity shock waves, (4.1)–(4.2) give 12 relations among the $16 n_{0i}, n_{ji}, \rho_j, \gamma_j$ parameters, leaving four arbitrary parameters. We choose as free parameters $S = y_1 + y_2$ and three n_{0i} among the four ones. First we find P from (4.2″): $P = [-A_2 \mp (A_2^2 - 4A_1 A_2)^{1/2}]/2A_1$ and we deduce y_1, y_2, and $\bar{n}_{ji}$ with the help of (4.2′), $i = 1, 2$. Second, as in Section 2, we obtain n_{j3} as a function of the free parameters:

$$\begin{aligned} n_{j3} = & -[(n_{00} + n_{03} y_j)(1 + 3y_j)(3 + y_j) + 2y_j n_{02}(1 + 3y_j) + 2y_j n_{01}(3 + y_j)] \\ & \times [3y_j(1 + y_j)^2]^{-1} \end{aligned} \tag{4.3}$$

where we use the last relation (4.1) for the determination of the unknown n_{0i}. Third, we find all the n_{ji} parameters: $n_{j0} = y_j n_{j3}$, $n_{ji} = \bar{n}_{ji} n_{j3}$ with $i = 1, 2$; finally, the third and the fourth relations (4.1) give the last four parameters ρ_j and γ_j, $j = 1, 2$.

The previous invariance property still arises, with $y \leftrightarrow y^{-1}$, $n_{00} \leftrightarrow n_{03}$, $n_{01} \leftrightarrow n_{02}$, from which we find $n_{j0} \leftrightarrow n_{j3}, n_{j1} \leftrightarrow n_{j2}$, $\rho \leftrightarrow \rho$, $\gamma \leftrightarrow -\gamma$, and $M(x, t) = N_0 + N_3 + 2(N_1 + N_2) \leftrightarrow M(-x, t)$.

The asymptotic $|x| \to \infty$ positivity constraints are more complicated than in Section 3: (i) If $\gamma_1\gamma_2 > 0$, necessarily $n_{0i} > 0$ and $n_{0i} + \sum n_{ji} > 0$; (ii) if $\gamma_1\gamma_2 < 0$, we must have $n_{oi} + n_{ji} > 0$, $j = 1, 2$. The positivity at $t = 0$ is obtained with conditions on the two $d_j > 0$ parameters of $u_i = d_i \exp(\gamma_j x + \rho_j t)$.

As for the similarity solutions of Section 3, the superposition of two similarity shock waves has in general different $|x| \to \infty$ limits for N_i, M,.... The macroscopic quantities M, J, and T have a jump between these limits (across the shock). At $t = 0$ or t small the profiles look like those in Section 3, but when t increases we observe deformations of the profiles. The half-lines $\xi_i = \gamma_i x + \rho_i t = 0$ divide the half-plane $t > 0$, x real into three subdomains in which the ξ_i have different signs (or when $\xi_i \to \mp\infty$, then $\Delta_i \to 1, 0$). For large x, t we find three different absolute Maxwellians (AM) (here three different constants). Two of them, $|x| \to \infty$, t finite, correspond to the upstream and downstream limits. The third AM is different. If $\rho_1\rho_2/\gamma_1\gamma_2 < 0$, it is the limit $|x|$ finite and $t \to \infty$ (equilibrium state). If $\rho_1\rho_2/\gamma_1\gamma_2 > 0$, the equilibrium AM coincides with one of the shock limits, while the third AM is reached for particular x, t intervals with lengths growing to infinity.

As in the 4-velocity model, we can construct $(1+1)$-dimensional shock wave corresponding to an almost infinite-strength shock. Then T is of the order of ε (ε arbitrarily small but fixed).

We begin with the $T = 0$ case, for which we have either $N_i \to \delta_{i3}$ or $N_i \to \delta_{i0}$ (which, using the above transform, can be reduced to the first case). Let us assume $n_{0i} = \delta_{i3}$: there exists a class of solutions $S = y_1 + y_2 < -6$, $-3 < y_1 < -1.66$, $y_2 < -3$, for which $\rho_j < 0$, $\gamma_j > 0$, and the upstream limit $x \to \infty$ corresponds to $T = 0$. For this class of solutions all n_{ij} are positive except n_{23} (but $1 + n_{23} > 0$) and n_{10}, n_{12} (but $n_{1i} + n_{2i} > 0$, $i = 0, 2$). However, these solutions are no good, because of negativity.

For $T \simeq \varepsilon$, with a small change of the upstream limits, $n_{03} = 1$, $n_{0i} = \varepsilon_i > 0$, $i \neq 3$ (ε_i arbitrarily small but fixed), choosing appropriate d_j values, we can obtain positive solutions. We define $\Delta_j = 1 + u_j$ and look at the densities N_0, N_2 for $t = 0$:

$$\Delta_1\Delta_2 N_i = \varepsilon_i + (n_{1i} + n_{2i}) + \varepsilon_i \sum u_j + n_{2i}u_1 + u_2(\varepsilon_i u_1 + n_{1i}), \qquad i = 0 \text{ and } 2 \tag{4.4}$$

Only the last term can be negative for $x > 0$ large. There exists $x_0 > 0$ such that $N_i > 0$ for $x < x_0$. For a complete positivity it is sufficient to choose

$$d_1 > \sup_i [-n_{1i}/\varepsilon_i \exp(-\gamma_1 x_0)] \tag{4.5}$$

For these almost infinite-strength shocks we find that the ratio of M across the shock is larger than 17. The limit 17 is obtained with the marginal degenerate case $y_1 = y_3 = -3$, where the two similarity components are equal with $n_{11} = n_{21} = 4$ and $M = 1 + 2N_1 = 17$.

For the other possibility, $n_{00} = 1$, $n_{0i} = \varepsilon_i$, $i \neq 0$, we use the $y \to y^{-1}$,..., previously defined transform. We find $-2/3 < S < 0$, $R > 17$, and $\rho_j < 0$, but $\gamma_j < 0$. As above, choosing appropriate d_2 constants, then N_i and M are positive everywhere. In Fig. 1b we present such an example with $n_{00} = 1$, $n_{01} = n_{02} = 5 \times 10^{-5}$, and $n_{03} = 25 \times 10^{-10}$. We observe for $t = 0$ or t small the usual shock profile of the similarity shock waves, but when t increases, we see the appearance of a plateau, which becomes larger and larger. We find three different AM, corresponding to three x, t subdomains: (i) $\xi_j = \gamma_j x + \rho_j t < 0$, the AM $(n_{0i} + \sum n_{ji})$ is both the upstream shock limit and the equilibrium state; (ii) $\xi_1 < 0$ and $\xi_2 > 0$, the AM $(n_{0i} + n_{1i})$ is the intermediate plateau; and (iii) $\xi_j > 0$, the AM is the downstream shock limit.

5. PERIODIC SOLUTIONS FOR THE 6-VELOCITY MODEL

Here $N_i = n_{0i} + n_i/\Delta + n_i^*/\Delta^*$, $\Delta = 1 + d\exp(\gamma x + \rho t)$, $n_i = n_{iR} + in_{iI}$, $\gamma = i\gamma_I$, $\rho = \rho_R + i\rho_I$.

5.1. Determination of the Solutions

Let us assume that the two similarity shock waves of Section 4 are complex conjugate. Then the compatibility condition (4.2)–(4.2′) becomes a relation between y_1 and $y_2 = y_1^*$. If we define $y = y_1 = |y| \exp(iz)$, this relation allows us to obtain the phase z from the modulus $|y|$. If, further, the superposition is periodic, then Re $\gamma = 0$ and we have one more relation than for the superposition of two real similarity shock waves. We shall have only three arbitrary parameters, chosen to be $|y|$ and two n_{0i} among the four ones.

Substituting the ansatz into (1.6), we find the complex similarity relations (3.1), (3.1′), coming from the coefficients of Δ^{-1}, Δ^{-2}:

$$\begin{gathered} n_1(n_0 + 3n_3) = -2n_0 n_3, \qquad n_2(3n_0 + n_3) = -2n_0 n_3, \\ -4\rho = n_1 + n_2 + 2(n_0 + n_3) \qquad \gamma(n_0 + n_3) = \rho(n_3 - n_0), \\ n_1(2\rho + \gamma) + n_{00} n_3 + n_{03} n_0 = n_{02} n_1 + n_{01} n_2 \qquad (5.1) \\ n_{01} n_{02} = n_{00} n_{03} \end{gathered}$$

There is in addition the compatibility condition between the two complex conjugate similarity components, coming from the vanishing of $|\Delta|^{-1}$:

$$\mathrm{Re}(n_1 n_2^* - n_0 n_3^*) = 0 \qquad (5.2)$$

We still define $y = n_0/n_3 = |y| \exp(iz)$, choose $|y|$ as a free parameter, and want to determine the phase z. For this we introduce the intermediate parameters

$$\bar{n}_1 = n_1/n_3 = -2y/(3+y), \qquad \bar{n}_2 = -2y/(3y+1) = n_2/n_3 \tag{5.3}$$

and find a cubic equation for $\cos z$:

$$\cos^3 z + [5(1+|y|^2)/3\,|y|] \cos^2 z + [(|y|^2 + |y|^{-2})/4 + 7/6] \cos z - (1+|y^2|)/3\,|y| = 0 \tag{5.2'}$$

We apply essentially the same method as in the previous sections. Assuming that $|y|$ and two n_{0i} are free parameters, the important step is the determination of n_3. Using the third and the fourth relations (5.1), we introduce other intermediate parameters, which are functions of $|y|$ alone:

$$\bar{\rho} = \rho/n_3 = -[\bar{n}_1 + \bar{n}_2 + 2(1+y)]/4, \qquad \bar{\gamma} = \gamma/n_3 = (1-y)/(1+y) \tag{5.4}$$

For periodic solutions we have $\operatorname{Re} \gamma = \operatorname{Re} \bar{\gamma} n_3 = 0$, which is the additional relation:

$$n_{3\mathrm{I}} = n_{3\mathrm{R}} \bar{\gamma}_\mathrm{R} / \bar{\gamma}_\mathrm{I} \tag{5.5}$$

We notice that (5.1) (5.2), (5.5) give 13 relations for the 16 real parameters n_i, n_{0i}, γ, and ρ, and consequently we have solutions with three arbitrary parameters. The last two relations (5.1), taking into account (5.2′)–(5.5), give three relations among the n_{0i} and $n_{3\mathrm{R}}$:

$$\begin{aligned}
n_{3\mathrm{R}}/\bar{\gamma}_\mathrm{I} &= (-\ |y| \sin z\, n_{03} + \bar{n}_{1\mathrm{I}} n_{02} + \bar{n}_{2\mathrm{I}} n_{01})/\lambda \\
&= (n_{00} + n_{03}\,|y| \cos z - \bar{n}_{1\mathrm{R}} n_{02} - \bar{n}_{2\mathrm{R}} n_{01})/\mu \\
n_{00} n_{03} &= n_{01} n_{02} \\
\lambda &= (2\bar{\rho}_\mathrm{I} + \bar{\gamma}_\mathrm{I})(\bar{n}_{1\mathrm{R}} \bar{\gamma}_\mathrm{I} - \bar{\gamma}_\mathrm{R} \bar{n}_{1\mathrm{I}}) + (2\bar{\rho}_\mathrm{R} + \bar{\gamma}_\mathrm{R})(\bar{n}_{1\mathrm{I}} \bar{\gamma}_\mathrm{I} + \bar{n}_{1\mathrm{R}} \bar{\gamma}_{1\mathrm{R}}) \\
\mu &= (2\bar{\rho}_\mathrm{R} + \bar{\gamma}_\mathrm{R})(-\bar{\gamma}_{1\mathrm{I}} \bar{n}_{1\mathrm{R}} + \bar{\gamma}_\mathrm{R} \bar{n}_{1\mathrm{I}}) + (2\bar{\rho}_\mathrm{I} + \bar{\gamma}_\mathrm{I})(\bar{n}_{1\mathrm{I}} \bar{\gamma}_\mathrm{I} + \bar{n}_{1\mathrm{R}} \bar{\gamma}_\mathrm{R})
\end{aligned} \tag{5.6}$$

From (5.6) and knowing $|y|$ and two n_{0i}, we determine both the other two n_{0i} and $n_{3\mathrm{R}}$. From $n_{3\mathrm{R}}$ and (5.5) we find $n_{3\mathrm{I}}$ and it follows that n_3 is known as a function of the arbitrary parameters.

Once n_3 is known, as in the previous sections, from the intermediate parameters we can reconstruct the original ones: $n_0 = yn_2$, $n_2 = \bar{n}_2 n_3$, $n_1 = \bar{n}_1 n_3$, $\rho = \bar{\rho} n_3$, $\gamma = \bar{\gamma} n_3$.

5.2. An Analytical Solution

The analytical determination of all the parameters n_{0i}, n_i, γ, and ρ is so complicated that we must use a computer for the explicit construction of the solutions. However, for $|y|=2$ [or $|y|=1/2$, because the compatibility relation (5.2′) is invariant when $|y| \to |y|^{-1}$] and $\cos z = 1/4$, an analytical solution exists. We choose n_{00} and n_{01} as the free parameters that determine completely this analytical solution. We obtain for the other n_{0i} and n_{3R}

$$n_{02} = n_{00}(7n_{00} + 4n_{01})/(4n_{01} - 2n_{00}), \qquad n_{03} = n_{01} n_{02}/n_{00}$$
$$n_{3R} = 3(n_{01}^2 + 5n_{00}^2/8 + 2n_{00}n_{03})/(n_{00} - 2n_{01}) \tag{5.7a}$$

Let us put $\zeta = \pm 1$ and for the densities N_i we define $\hat{n} = (N_i - n_{0i})/n_{3R}$ and get

$$\hat{n}_0 = 8/3 \operatorname{Re}(1 + i\zeta \sqrt{15}/3)\, \Delta^{-1}, \qquad \hat{n}_1 = -2 \operatorname{Re}(1 + i\zeta \sqrt{15}/9)\, \Delta^{-1}$$
$$\hat{n}_2 = -4/3 \operatorname{Re}(1 - i\zeta \sqrt{15}/15)\, \Delta^{-1}, \qquad \hat{n}_3 = 2 \operatorname{Re}(1 - i\zeta \sqrt{15}/9)\, \Delta^{-1}$$
$$\Delta = 1 + d \exp n_{3R}[-3t/4 + i\zeta \sqrt{15}/5\, (x - 3t/4)] \tag{5.7b}$$

As an example, if we choose for the free parameters $0 < n_{00}/2 < n_{01}$, then from (5.7a), (5.7b), we find $n_{3R} < 0$, and when $t \to \infty$, $\Delta^{-1} \to 0$ and the densities $N_i \to n_{0i} > 0$.

5.3. Properties of the Periodic Solutions

For the asymptotic $t \to \infty$ positivity, we must either have $n_{0i} > 0$ if $\rho_R > 0$ or $n_{0i} + 2n_{iR} > 0$ if $\rho_R < 0$. When these limits are different from zero, it is clear that there exists t_0 such that $N_i > 0$ for $t > t_0$. We remark that this property, true for the periodic solutions, does not necessarily hold for a superposition of real similarity components. For instance, let us assume $\rho_R > 0$ and consider first the periodic solutions $\Delta^{-1} \to 0$ when $t \to \infty$ and becomes negligible for large fixed t and x finite or infinite. On the contrary when Δ, ρ, and γ are real and t is large and finite, then $\Delta \to 1$ when $\gamma x \to -\infty$. However, the physical problem is to choose t_0 fixed, for instance, $t_0 = 0$, and to find the conditions on d such that positivity is ensured. This was previously done for the Broadwell model,[1] establishing appropriate lower bounds on $N_i\, |\Delta|^2\, n_{0i}^{-1}$. For instance, when $\rho_R > 0$, then for $t \geq 0$ and all x values $N_i > 0$ is satisfied if, for $i = 0, 1, 2, 3$,

$$|d| > \sup_i X_i, \qquad X_i = 1 + |n_i|/n_{0i} + [(1 + |n_i|/n_{0i})^2 - 1 + 2\, |n_{1R}|/n_{0i}]^{1/2} \tag{5.8}$$

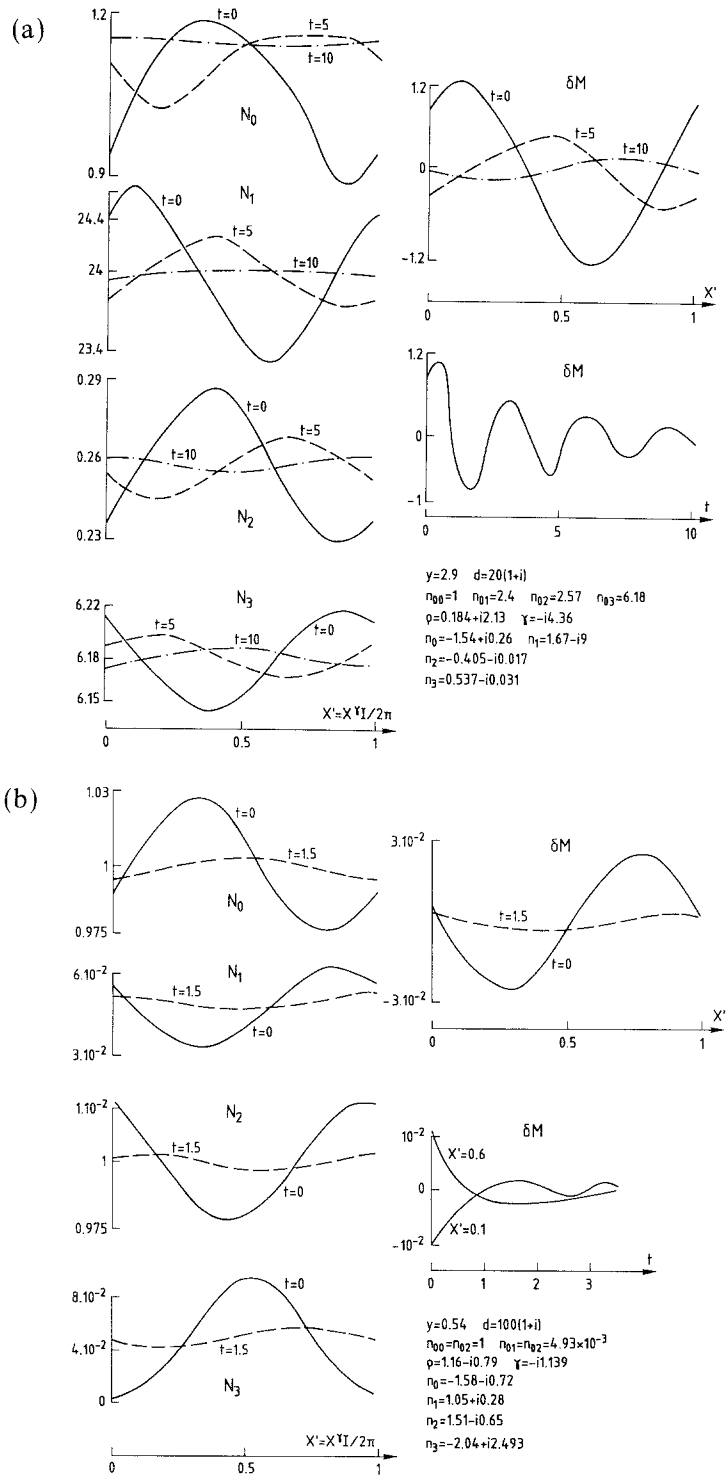

Fig. 2. Periodic solutions with (a) $J_{eq} \neq 0$, (b) $J_{eq} = 0$.

For instance, for the analytical solution (5.7) with $n_{00}=n_{01}=1$ and $\rho_R>0$ we have verified that $d=34(1+i)$ leads to positive N_i for $t\geqslant 0$ and all x values.

We write down the total mass $M=N_0+N_3+2(N_1+N_2)$ and momentum $J=N_0-N_3+N_1-N_2$, and look at their large-time behavior: $M=M_{eq}+\delta M$, $J=J_{eq}+\delta J$. For large t, δM and δJ are small perturbations, while if $\rho_R>0$, $M_{eq}=n_{00}+n_{03}+2(n_{01}+n_{02})$ and $J_{eq}=n_{00}-n_{03}+n_{01}-n_{02}$ are the equilibrium states (if $\rho_R<0$, we must add $2\operatorname{Re} n_i$ to each n_{0i}). In both cases we have

$$\begin{aligned}\delta M &\simeq 2A_M e^{-|\rho_R|t}\cos(\gamma_I x+\rho_I t+\phi_M)\\ \delta_J &\simeq 2A_J e^{-|\rho_R|t}\cos(\gamma_I x+\rho_I t+\phi_J)\end{aligned}\tag{5.9}$$

with A_M and A_J positive constants and ϕ_M and ϕ_J constant phase factors.

Clearly δ_M and δ_J represent propagating ($\rho_I\neq 0$) and damped ($\rho_R\neq 0$) plane waves. If the ratio of the absorption coefficient ρ_R to the frequency ρ_I is small, we have many effective oscillations. In Fig. 2a, for $|y|=2.9$, $n_{00}=1, n_{01}=2.4$, we present such a solution with $|\rho_R/\rho_I|\simeq 0.08$ and we observe many oscillations for δM.

If $J_{eq}=0$ and $\rho_R>0$, we have in addition $n_{00}-n_{03}+n_{01}-n_{02}=0$; (5.6) becomes $n_{02}=n_{00}$ and

$$n_{03}=n_{01}=(\bar{n}_{1R}-1+\mu\bar{n}_{1I}/\lambda)/[|y|\cos z-\bar{n}_{2R}+\mu(|y|\sin z-\bar{n}_{2I})/\lambda]\tag{5.10}$$

In that case we have two free parameters $|y|$ and n_{00}. From a numerical analysis we find that $|\rho_R/\rho_I|$ is not small. In Fig. 2b we report an example with $|y|=0.54$, $n_{00}=1$, and $|\rho_R/\rho_I|\simeq 1.46$ and we observe few oscillations because the wave is strongly damped.

6. MORE GENERAL PLANAR VELOCITY MODELS

For the $2r$-velocity models with $r\geqslant 4$, the new fact in the algebraic structure of the discrete models (1.3) is the existence of independent (not proportional) collision terms: two for $r=4, 5$; three for $r=6, 7$;.... Consequently, the number of compatibility conditions increases for a superposition of two similarity shock waves. For $r=4$, the 8-velocity model, counting the number of parameters (n_{0i}, n_{ji}, γ_i, ρ_i) versus the number of real relations, we find 18 relations and 19 parameters. From an analytical study followed by a numerical analysis, we have not found superposition of two similarity shock waves satisfying all the relations. For $r=5$ and the superposition of two similarity solutions, the number of parameters is 22, while the number of relations is 20. Always for a superposition of two real similarity solutions, let us call P the number or parameters and R the number of relations. Then from a crude counting argument we find: (i) $P = 6q + 7$ and $R = 8q + 2$ if $r = 2Q$ is even, (ii) $P = 6q + 10$ and $R = 8q + 4$ if $r = 2q + 1$ is odd. In both cases the number of relations increases more than the number of parameters. We find $P \leqslant R$ for $r = 6$ if $r = 2q$ and for $r = 7$ if $r = 2q + 1$. Finally, we notice that for the periodic solutions we have one more relation.

7. LIMITS FOR THE MEAN FREE PATH GOING TO ZERO

In (1.5), (1.6) we divide the collision term by the mean free path ε and look[9] at the limit $\varepsilon \to 0$. The exact ε-dependent solutions are found with the changes $t, x \to t/\varepsilon,\ x/\varepsilon$ and the limits are the constant AM: one, the equilibrium state, for the periodic solutions $(t \neq 0)$, two for the planar shock waves $(\xi = \gamma x + \rho t \to \mp \infty)$, and three for the nonplanar ones (three subdomains where the $\xi_i = \gamma_i x + \rho_i t$ have well-defined signs, as is discussed in Section 4). An ε expansion around $\varepsilon = 0$ is not possible, while a natural parameter is $\exp(-1/\varepsilon)$.

8. CONCLUSION

We can collect $(1+1)$-dimensional exact solution results for the 2-velocity models on a line, 6-velocity in three-dimensional space, and 4- and 6-velocity in a plane. These exact solutions are linear superpositions of similarity shock waves (real or complex conjugate). For more general planar models, we notice that when the number of velocities increases, then the number of relations to be satisfied by the parameters increases more than the number of parameters.

Perhaps the most interesting physical property of these discrete models is to provide explicit solutions for the strong shocks. For the exact solutions, the positivity problem is not trivial; however, we have succeeded in the construction of positive solutions with an almost infinite-strength shock.

Recall that positivity is also difficult to handle for the shock solutions of the continuous Boltzmann equation (in particular for the infinite-Mach shock[9]). In general this problem is solved by taking into account only a finite number of moments of the distribution, which is another discretization. Here the strong-shock solution found by Broadwell[3] has been generalized in $1 + 1$ dimensions for all the above discrete models.

The periodic solutions represent propagating and damped waves; we have found examples with many oscillations and others with few oscillations.

Also recall that the main difficulties in the hydrodynamic limits of the kinetic solutions come from the existence of shock layers, boundary layers, and initial layers. We remark that these exact solutions provide explicit examples of problems related to layers.

If we except the 2-velocity models, all the others have generalizations in at least two spatial dimensions. For instance, for the 4-velocity model [(1.5) in 1 + 1 dimensions] we have obtained a class of (2 + 1)-dimensional exact solutions, which depend on six parameters, and at present we are trying to extract a subclass of positive solutions.

APPENDIX. POSSIBLE MULTIEXPONENTIAL RATIONAL SOLUTIONS

Our aim is to show that the existence of two independent linear differential relations such as those of (1.5) and (1.6) give strong restrictions on the possible rational solutions with independent ($\gamma_i \rho_j \neq \gamma_j \rho_i$) exponential $u_j = d_j \exp(\gamma_j x + \rho_j t)$ variables. The single-exponential rational solutions compatible with the quadratic nonlinearity are the similarity shock waves with denominators $\Delta = 1 + u_j$. Any linear superposition leading to factorized denominators $\Delta = \Pi\ (1 + u_j)$ is possible at the linear level. But other single-exponential rational solutions and their linear superposition, leading to other factorizations of Δ, satisfy also the two linear relations. For instance, solutions with denominators of the type $\Delta^p = (1+u)^p$, $p = 2, 3,...$, and their linear superposition are rational solutions satisfying both linear relations. However, they are not compatible with the nonlinearity, which requires $2p = p + 1$ or $p = 1$. Keeping in mind the necessary limitations of a study at the linear level, we disregard factorized denominators. We look at the restrictions for possible rational solutions that are not necessarily linear superpositions. For instance, $\Delta = 1 + u_1 + u_2$, which is possible for the peculiar model[6] with one conservation law, is not possible for models with two independent linear relations.[1] This is the type of result that we want to enlarge.

Let us assume that three rational N_i defined by

$$N_i = n_i + M_i/\Delta, \qquad \Delta = 1 + \sum_1^p u_j, \qquad M_i = n_{0i} + \sum_1^{p-1} n_{ji} u_j \tag{A.1a}$$

$$\gamma_j \rho_i \neq \gamma_i \rho_j \quad \forall i \neq j, \qquad u_j \neq \text{const} \quad \forall j \tag{A.1b}$$

satisfy the two linear independent differential relations

$$(N_0 + b_m N_m)_t + (N_0 + c_m N_m)_x = 0, \qquad m = 1, 2 \tag{A.2a}$$

$$b_m \neq c_m, \qquad b_1 c_2 \neq b_2 c_1 \tag{A.2b}$$

We rewrite (A.2a) using (A.1a):

$$M_0(\Delta_t + \Delta_x) + N_1(b_m \Delta_t + c_m \Delta_x) - \Delta(M_{0t} + M_{0x} + b_m M_{mt} + c_m M_{mx}) = 0 \tag{A.3}$$

In a two-dimensional space, the relations (A.1b) are essential. We recall[1] that for $p = 2$, this relation is violated while for $p = 3$; the only possibility is the factorized $\Delta = (1 + u_1)(1 + u_2)$.

Here we push the analysis up to $p = 4$. Substituting (A.1a) into (A.3), we have ten $(u_j, u_k u_{k'})$ terms, leading to ten distinct relations if none of these terms is proportional and less otherwise. We establish a lemma useful for the simplest cases.

Lemma A1. Assuming that for i fixed, $i = 1, 2, 3$, the three terms u_i, u_4, and $u_i u_4$ in the relation (A.3) are not proportional to any one of the seven others, then necessarily

$$\gamma_4 \rho_i = \rho_4 \gamma_i$$

For the proof we write that the coefficients of the three terms u_i, u_4, and $u_i u_4$ are zero;

$$(\rho_i + \gamma_i)(n_{00} - n_{i0}) = (b_m \rho_i + c_m \gamma_i)(n_{im} - n_{0m}) \tag{A.4a}$$

$$n_{00}(\rho_4 + \gamma_4) + n_{0m}(b_m \rho_4 + c_m \gamma_4) = 0 \tag{A.4b}$$

$$n_{i0}(\rho_4 - \rho_i + \gamma_4 - \gamma_i) + n_{im}[b_m(\rho_4 - \rho_i) + c_m(\gamma_4 - \gamma_i)] = 0 \tag{A.4c}$$

From (A.4a)–(A.4c) we deduce

$$n_{00}/n_{i0} = (b_m \rho_4 + c_m \gamma_4)/[b_m(\rho_4 - \rho_i) + c_m(\gamma_4 - \gamma_i)], \qquad m = 1, 2 \tag{A.5a}$$

or equivalently $(b_1 c_2 - b_2 c_1)(\gamma_4 \rho_i - \rho_4 \gamma_i) = 0$. The first factor is different from zero, due to (A.2b), and so the second is zero.

We discuss the different possibilities:

(i) None of the $(u_j, u_k u_{k'})$ are proportional.

(ii) Only two terms are proportional: either $(\text{ii})_1$ one u_j and one $u_k u_{k'}$, for instance: $u_3 = \text{const} \cdot u_1 u_2$, or $(\text{ii})_2$ two $u_k u_{k'}$ are proportional; the only possibility compatible with the assumptions (A.1b) is $u_1 u_2 = \text{const} \cdot u_3 u_4$.

(iii) For more $(u_j, u_k u_{k'})$ terms proportional, from (A.1b) they cannot be of the $(\text{ii})_2$ type alone or a mixture of $(\text{ii})_1$ and $(\text{ii})_2$, but only two $(\text{ii})_1$ relations, for instance, $u_3 = C_1 u_1 u_2$, $u_4 = C_2 u_1 u_3$, the C_i being constants.

In case (i), we apply the lemma for $i = 1, 2, 3$ and find $\gamma_i/\rho_i = \text{const}$ independent of $i = 1, 2, 3, 4$. This result contradicts the assumption (A.1b). In case $(\text{ii})_1$, with $\rho_3 = \rho_1 + \rho_2, \gamma_3 = \cdots$, we apply the lemma for $i = 1, 2$ and we still find the same constant for the ratio γ_i/ρ_i and all four values. In case (iii), we cannot apply the lemma because u_4 is proportional to $u_1 u_3$. In principle we have eight relations, but it turns out that the vanishing of the coefficient of $u_2 u_3$ gives an identity.

Case $u_3 = C_1 u_1 u_2, u_4 = C_2 u_1 u_3$. We have seven independent relations coming from (A.3): $i = 1, 2$ in (A.4a); $i = 1, 2, 3$ in (A.4c); and two others. The first one includes terms proportional to both $u_1 u_2$ and u_3, while the second contains both u_4 and $u_1 u_3$. After a tedious calculation we find

$$n_{20}/2 = n_{30} = n_{10}(b_m \rho_1 + c_m \gamma_1)/[b_m(\rho_1 + \rho_2) + c_m(\gamma_1 + \gamma_2)]$$

and the interesting result

$$n_{00}/n_{10} = [b_m(2\rho_1 + \rho_2) + c_m(2\gamma_1 + \gamma_2)]/[b_m(\rho_1 + \rho_2) + c_m(\gamma_1 + \gamma_2)], \quad m = 1, 2 \quad \text{(A.6)}$$

From (A.6) and taking into account (A.2b) we find $\rho_1 \gamma_2 = \gamma_1 \rho_2$ and with the relations between the u_j, we get that (A.1b) is not satisfied for $i = 1, 2, 3, 4$.

If we consider higher p values, the number of subcases involving terms $(u_j, u_k u_{k'})$ increases, too. From a partial analysis of the $p = 5$ case we have not found solutions involving several exponentials and with Δ not factorized.

REFERENCES

1. H. Cornille, *J. Phys. A* **20**:1973 (1987); *J. Math. Phys.* **28**:1567 (1987); Saclay PhT 86/145.
2. Illner, *Mat. Meth. Appl. Sci.* **1**:187 (1979); *Commun. Math. Phys.* **94**:353 (1984).
3. J. E. Broadwell, *Phys. Fluids* **7**:1243 (1964); L. Tartar, Séminaire Goulaouic-Schwartz NI (1975–1976); J. T. Beale, *Commun. Math. Phys.* **94**:341 (1968).
4. Gatignol, *Lecture Notes in Physics*, Vol. 36 (1976).
5. D. McKean *Commun. Pure Appl. Math.* **8**:435 (1975); M. Shinbrot, in *Oberwelfach Conference*, D. Pach and M. Heunzert, eds. (1979), p. 11.
6. T. W. Ruijgrok and T. T. Wu, *Physica* **113A**:401 (1982).
7. A. V. Bobylev, Mathematics Congress, Warsaw (1985) (Book Abstract B 29); J. Wick, *Math. Math. Meth. Appl. Sci.* **6**:515 (1984); G. Dukek and T. F. Nonnenmacher, *Physica* **135A**:167 (1978); T. Platkowski, *J. Mec. Théor. Appl.* **4**:555 (1985).

8. S. Harris, *Phys. Fluids* **9**:1328 (1966); I. Hardy and Y. Pomeau, *J. Math. Phys.* **13**:1046 (1972); H. Cabannes, *J. Mec. Theor. Appl.* **17**:1 (1979); *Oberwelfach Conference* (1979); Lecture Notes, University of California, Berkeley (1980).
9. R. Caflish, in *Nonequilibrium Phenomena I*, (J. L. Lebowitz, ed.) (1983), p. 193; *Commun. Pure Appl. Math.* **38**:529 (1985); R. E. Caflish and G. C. Papanicolaou, *Commun. Pure Appl. Math.* **32**:589 (1979); R. E. Caflish and T. P. Liu, in *15th Symposium on Rarefied Gas Dynamics*, V. Boffi and C. Cercignani, eds. (B. G. Teubner, Stuttgart, 1986).